NEWS ON THE TRANSVERSE WALL IMPEDANCE AT LOW FREQUENCIES

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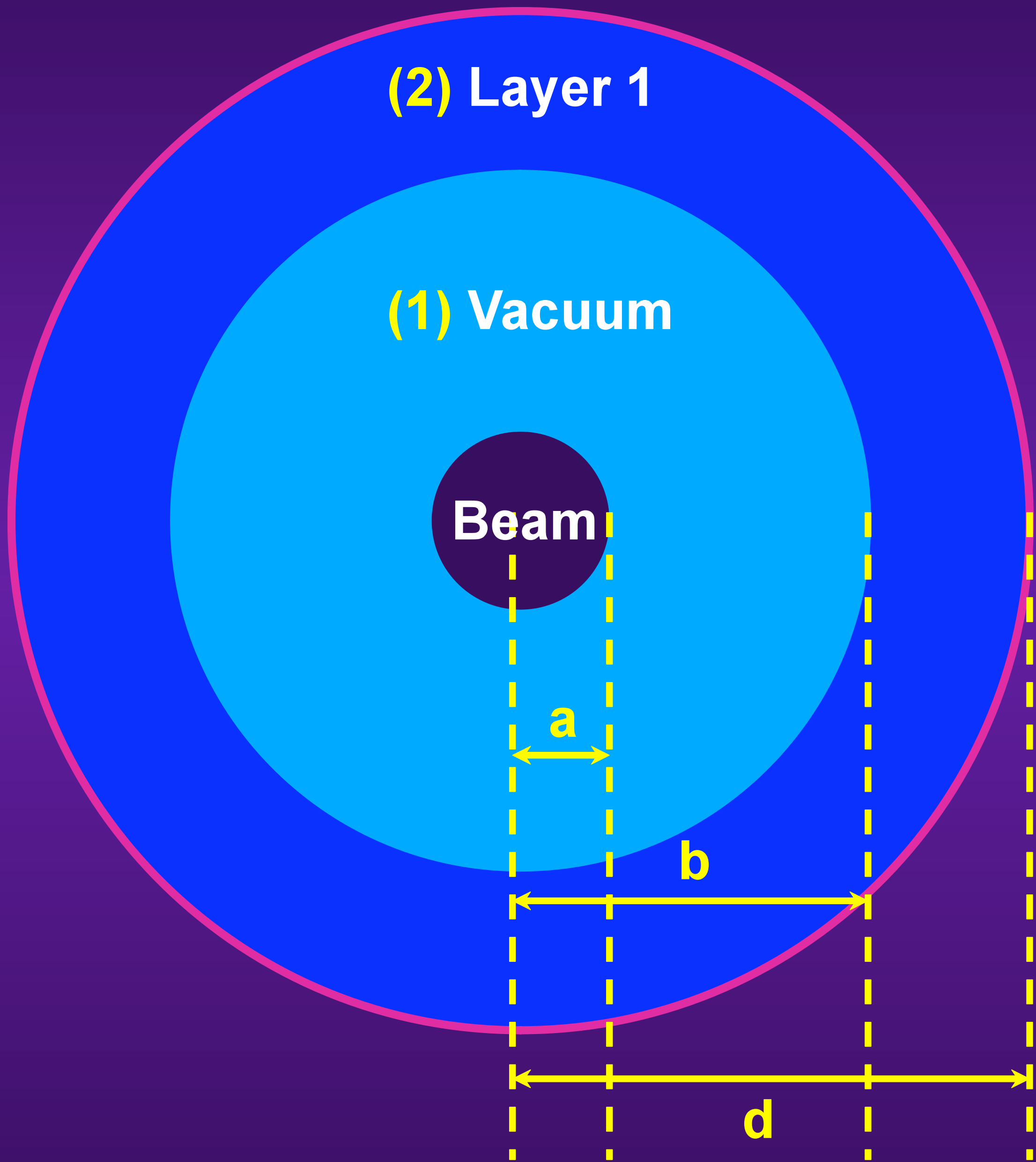
- **Reminder: Review of Zotter's formalism ➔ http://care-hhh.web.cern.ch/care-hhh/Collective%20Effects-GSI-March-2006/default.html**
- **Introduction**
- **Where does the low-frequency inductive impedance come from?**
- **Impedance in the (quasi-) static case**
- **Why the result from Burov-Lebedev2002 is close to ours whereas they consider only TM modes?**
- **Approximate formula for a LHC (graphite) collimator**
- **Conclusion and future work**

INTRODUCTION (1/4)

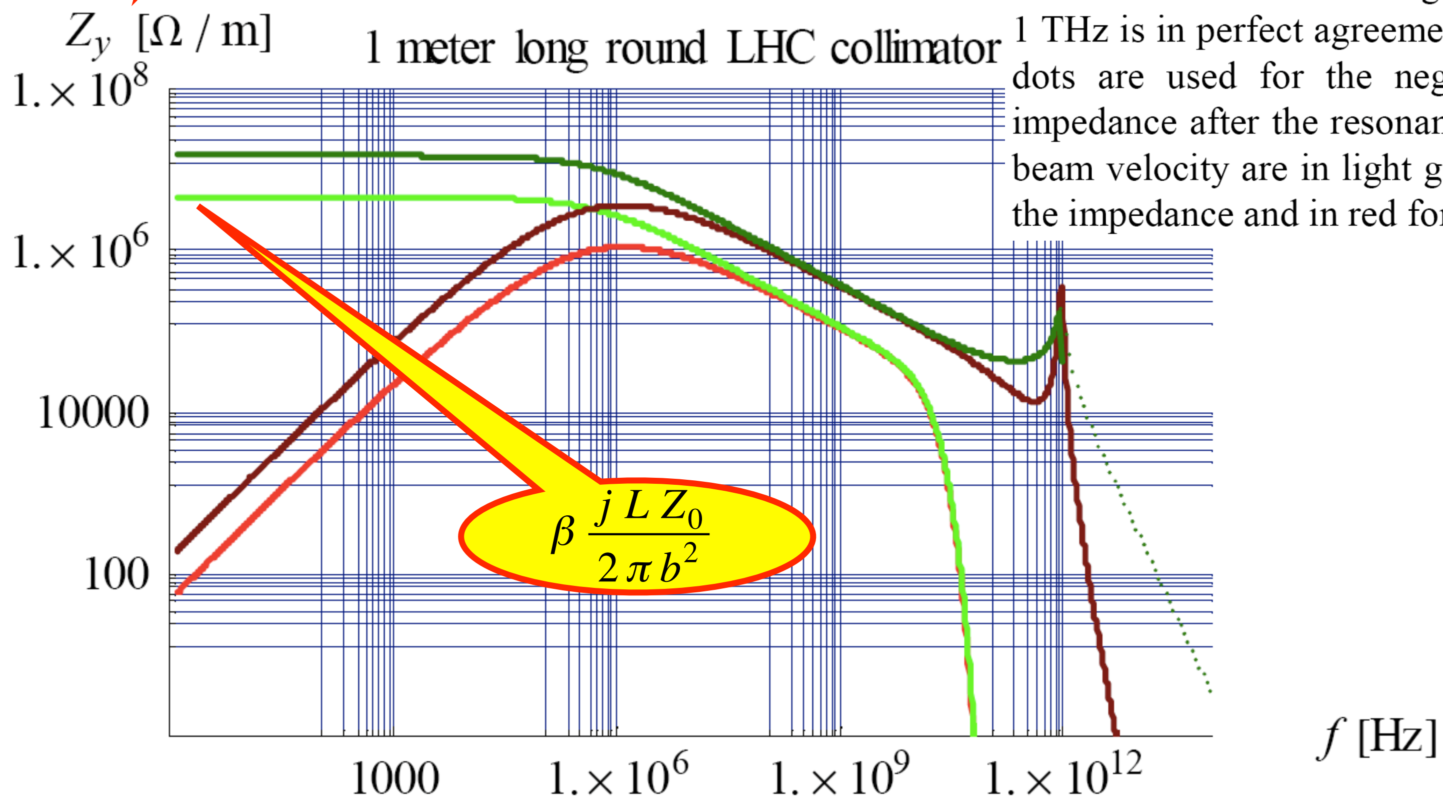


INTRODUCTION (2/4)

"RW" impedance

PAC05 paper

Figure 2: Zotter's results for the LHC, where $\gamma = 7462.69$ and $\beta = 1$, and for the CERN PSB, where $\gamma = 1.05$ and $\beta = 0.3$, to see the effect of a lower beam velocity. An AC conductivity is assumed here, $\sigma_{\mathrm{AC}} = \sigma_{\mathrm{DC}} / \left(1 + j\,\omega\,\tau\right)$, where $\tau \approx 0.8$ ps is the relaxation time. The high-frequency resonance near 1 THz is in perfect agreement with Bane's results [8] (the dots are used for the negative imaginary part of the impedance after the resonance). The curves for the lower beam velocity are in light green for the imaginary part of the impedance and in red for the real part.

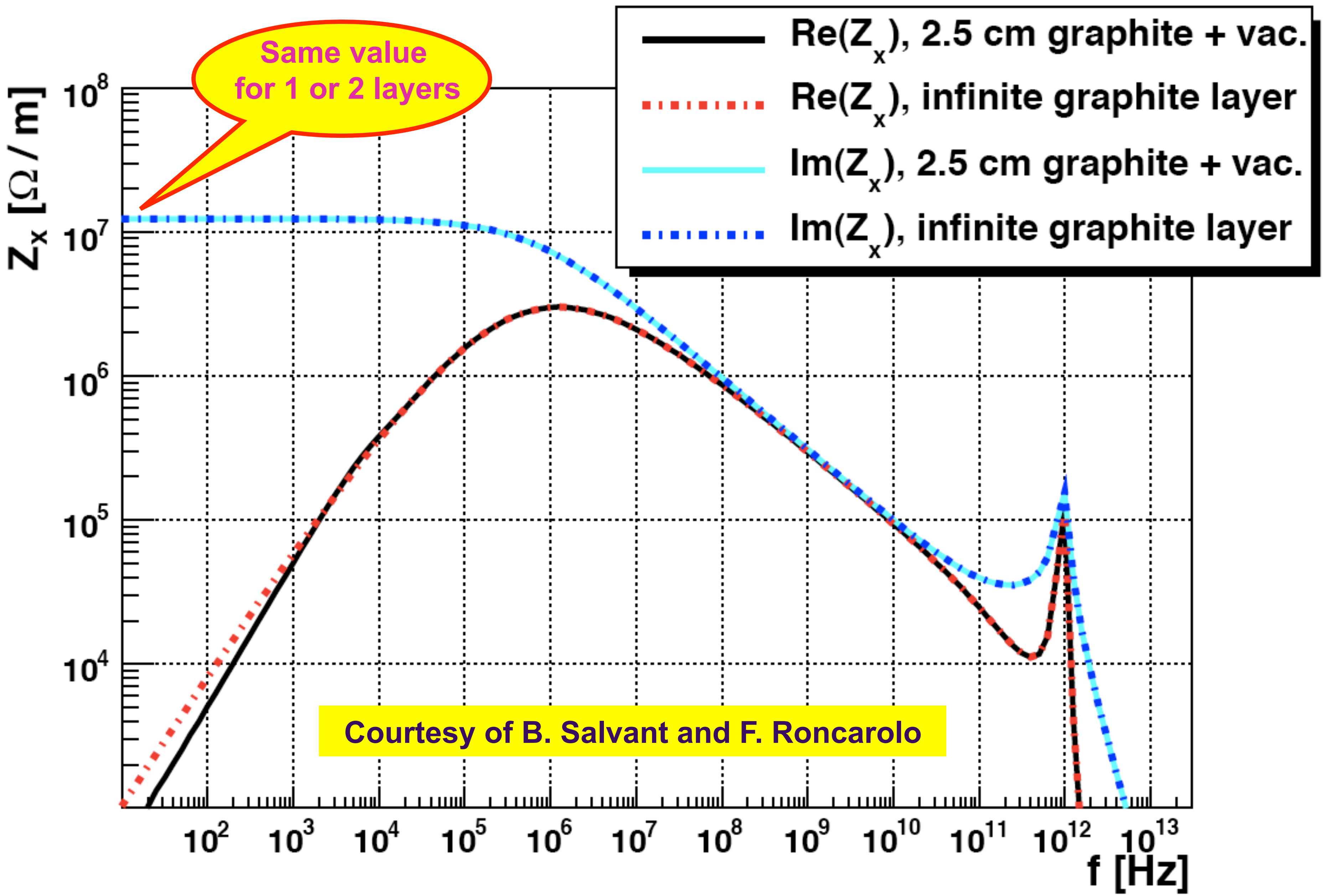


$b = 2$ mm

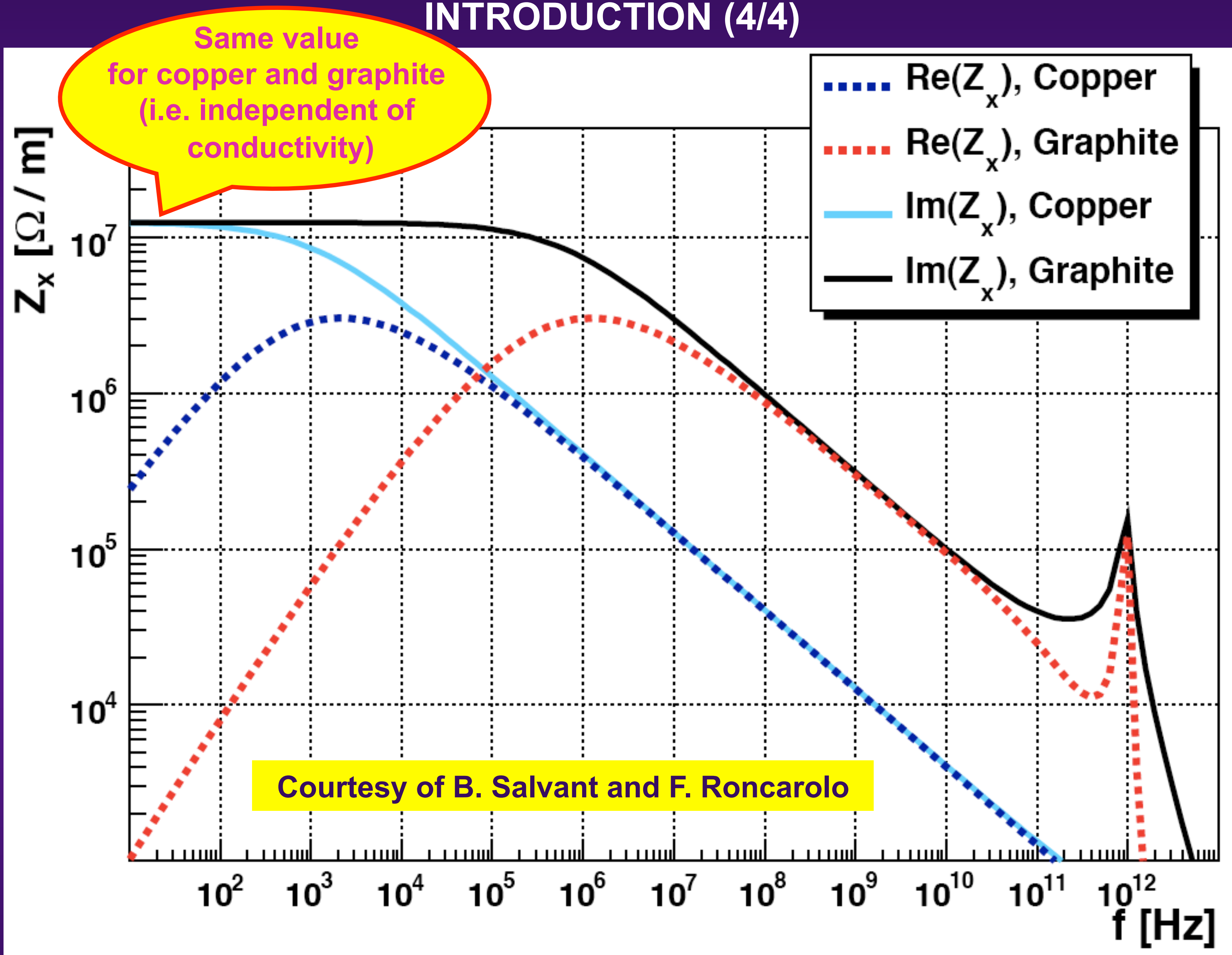
$d_C = \infty$

$\rho_C = 10\ \mu\Omega\mathrm{m}$

INTRODUCTION (3/4)



INTRODUCTION (4/4)



WHERE DOES THE LOW-FREQUENCY INDUCTIVE IMPEDANCE COME FROM? (4 slides)

- **To compute the transverse "Resistive-Wall" (RW) impedance from field matching, one subtracts from the "Total" impedance the so-called "Transverse Space-Charge" (SC) impedance, given by (for a round beam in a round pipe and below a certain frequency)**

$$Z_x^{SC} = - j \, \frac{L \, Z_0}{2 \pi \beta \gamma^2} \left(\frac{1}{a^2} - \frac{1}{b^2} \right)$$

Incoherent part (from the beam)

Coherent part (from the pipe)

- **Let's consider now only the coherent part of the SC impedance, as the incoherent one is of no interest here**

From electric images

From ac magnetic images

=> $$Z_x^{SC,coh} = j \, \frac{L \, Z_0}{2 \pi \beta \gamma^2 \, b^2} = j \, \frac{L \, Z_0}{2 \pi \beta \, b^2} \left(1 - \beta^2 \right)$$

- **Note that there is no dc magnetic images as no magnet poles are considered here (it is as if they were at infinity)**

=> $$Z_x^{Total}(f) = Z_x^{SC,coh} + Z_x^{RW}(f)$$

=> $$Z_x^{RW}(f \rightarrow 0) = Z_x^{Total}(f \rightarrow 0) - Z_x^{SC,coh}$$

- **When f ➔ 0 one should have** $$Z_x^{Total}(f \rightarrow 0) = j\,\frac{L\,Z_0}{2\pi\,\beta\,b^2}$$

Only electric images should contribute as there are no ac magnetic images when f ➔ 0 (= result from the static case)

=> One should have

$$Z_x^{RW}(f \rightarrow 0) = j\,\frac{L\,Z_0}{2\pi\,\beta\,b^2} - j\,\frac{L\,Z_0}{2\pi\,\beta\,b^2}\left(1-\beta^2\right) = \beta\,j\,\frac{L\,Z_0}{2\pi\,b^2}$$

- … and this is exactly what we have from the general theory using Zotter's formalism. In fact we have

$$Z_x^{RW}\left(f \rightarrow 0 \right) = \beta \, \frac{j\, L\, Z_0}{\pi\, b^2} \times \frac{1}{1 + \dfrac{1}{\mu_r}}$$

Therefore, if $\mu_r = 1$ => $$Z_x^{RW}\left(f \rightarrow 0 \right) = \beta\, j\, \frac{L\, Z_0}{2\pi\, b^2}$$

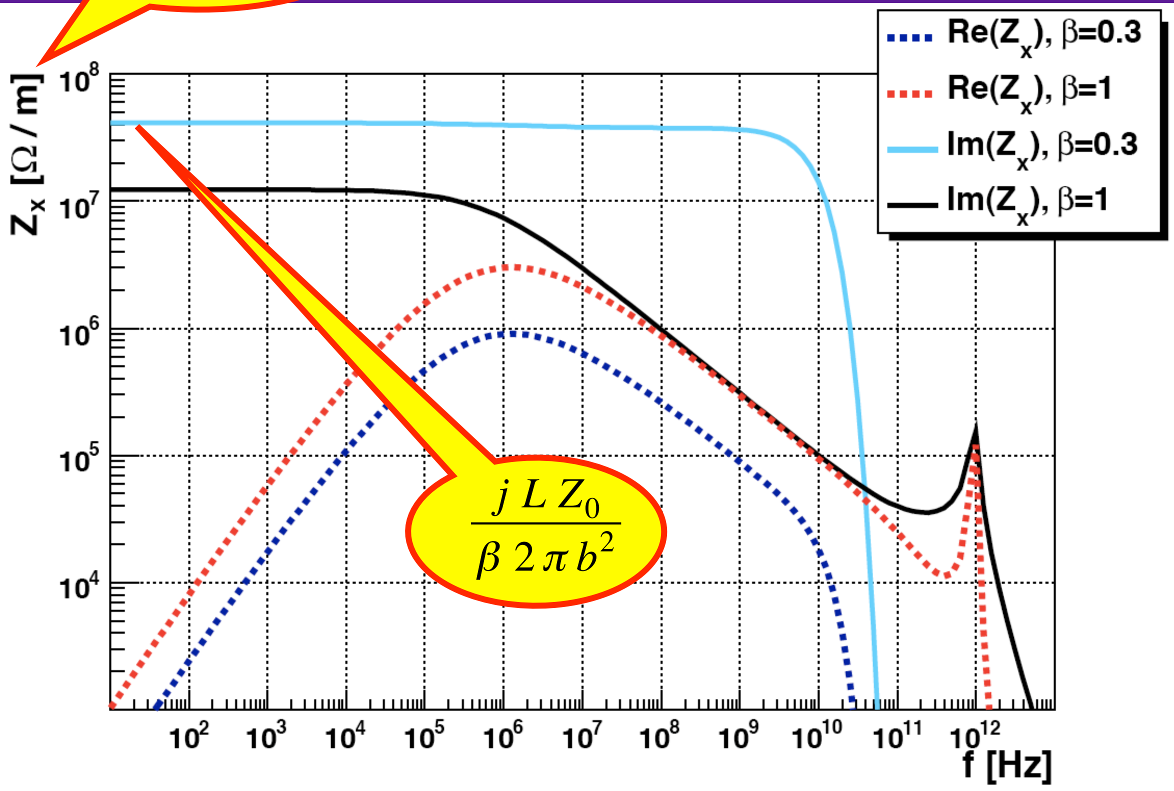
- … but this value appeared just because we removed from the total impedance the SC impedance (with the contribution from the ac magnetic images) which we should not do!

=> We believe it is better to define the "wall impedance" (removing only the incoherent part from the beam) instead of the "resistive-wall impedance"

$$Z_x^{Wall}\left(f \rightarrow 0\right) = \beta\, j\, \frac{L\, Z_0}{2\pi\, b^2} + j\, \frac{L\, Z_0}{2\pi\, \beta\, b^2}\left(1-\beta^2\right) = j\, \frac{L\, Z_0}{2\pi\, \beta\, b^2}$$

Only contribution from the electric images

"Wall" impedance

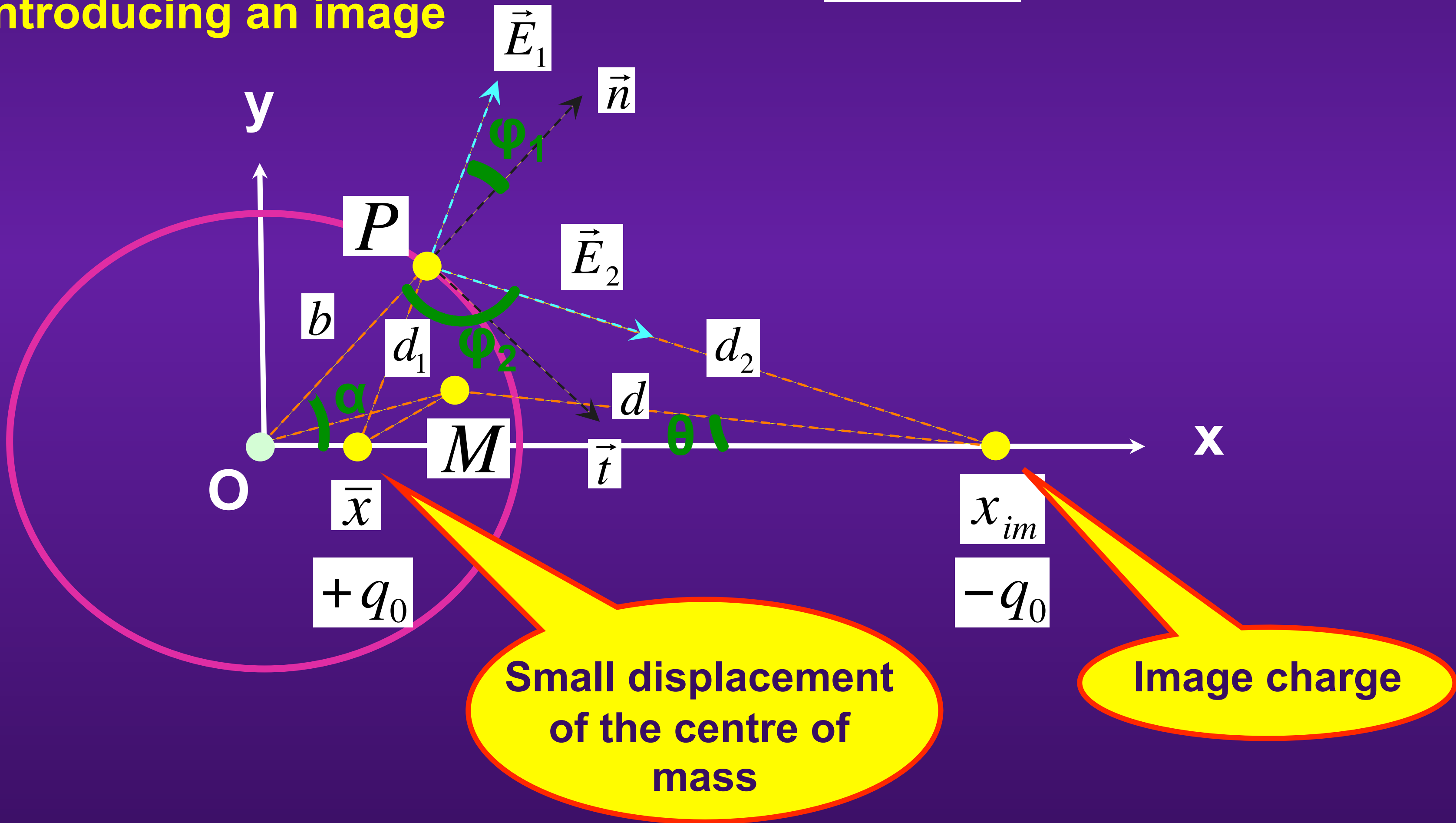


$$\frac{j\, L\, Z_0}{\beta\, 2\pi\, b^2}$$

Courtesy of B. Salvant and F. Roncarolo

IMPEDANCE IN THE (QUASI-) STATIC CASE (1/9)

- **Effect of the images (i.e. the wall) in the case of a circular beam off-axis in a (Perfectly Conducting, PC) circular beam pipe**
  - **The boundary condition on a PC ($E_t = 0$) is satisfied by introducing an image**



IMPEDANCE IN THE (QUASI-) STATIC CASE (2/9)

- $$E_1 = \frac{\lambda}{2\,\pi\,\varepsilon_0\,d_1}$$
  $$E_2 = \frac{\lambda}{2\,\pi\,\varepsilon_0\,d_2}$$

Line density

- $$E_{1t} = -\,E_1 \sin\varphi_1$$
  $$E_{2t} = E_2 \sin\left(\pi - \varphi_2\right) = E_2 \sin\varphi_2$$

- $$E_t = 0 \quad \Rightarrow \quad E_{1t} + E_{2t} = 0$$ => $$\frac{\sin\varphi_1}{d_1} = \frac{\sin\varphi_2}{d_2}$$

- **Relations in a triangle**

$$\frac{x_{im}}{\sin\varphi_2} = \frac{d_2}{\sin\alpha}$$

$$d_1^2 = \bar{x}^2 + b^2 - 2\,\bar{x}\,b\cos\alpha$$

$$\frac{\bar{x}}{\sin\varphi_1} = \frac{d_1}{\sin\alpha}$$

$$d_2^2 = x_{im}^2 + b^2 - 2\,x_{im}\,b\cos\alpha$$

IMPEDANCE IN THE (QUASI-) STATIC CASE (3/9)

=> $$\frac{\bar{x}}{d_1^2} = \frac{x_{im}}{d_2^2}$$ and $$\left(\frac{x_{im}}{\bar{x}}\right)^2 - \left[1 + \left(\frac{b}{\bar{x}}\right)^2\right]\frac{x_{im}}{\bar{x}} + \left(\frac{b}{\bar{x}}\right)^2 = 0$$

=> $$x_{im} = \bar{x}$$ or $$x_{im} = \frac{b^2}{\bar{x}}$$

=> The correct position for the image charge ($-q_0$) is $$x_{im} = \frac{b^2}{\bar{x}}$$

- To compute the image electric force, we place a witness line charge (λ) at point M (x,y). The electric force is

$$\frac{F_x^{ele}}{e} = E_x = \frac{\lambda}{2\,\pi\,\varepsilon_0\,d}\cos\vartheta$$

IMPEDANCE IN THE (QUASI-) STATIC CASE (4/9)

- $\cos\vartheta = \frac{x_{im} - x}{d}$ **=>** $\frac{\cos\vartheta}{d} = \frac{x_{im} - x}{d^2} = \frac{x_{im} - x}{\left(x_{im} - x\right)^2 + y^2}$

With $x_{im} = \frac{b^2}{\bar{x}}$ **=>** $\frac{F_x^{ele}}{e} = \frac{\lambda}{2\,\pi\,\varepsilon_0}\,\frac{\frac{b^2}{\bar{x}} - x}{\left(\frac{b^2}{\bar{x}} - x\right)^2 + y^2}$

=> $\frac{F_x^{ele}}{e} \approx \frac{\lambda}{2\,\pi\,\varepsilon_0}\,\frac{\bar{x}}{b^2}$ for $\bar{x} << \mathrm{b}$

With only contribution from the electric images

=> $Z_x^{Wall}\left(f \sim 0\right) = j\,\frac{L\,Z_0}{2\pi\,\beta\,b^2}$

IMPEDANCE IN THE (QUASI-) STATIC CASE (5/9)

- **This result is valid for ANY conductor in fact, i.e. it is not valid only for a PC, as after a very short time (relaxation time) all the induced electric charges move to the surface**

  - **Maxwell equations valid in homogeneous, isotropic, continuous media**

$$div\ \vec{E} = \frac{\rho}{\varepsilon} \qquad div\ \vec{H} = 0 \qquad \overrightarrow{rot}\ \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \qquad \overrightarrow{rot}\ \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

with $\vec{B} = \mu\ \vec{H}$ $\quad \varepsilon = \varepsilon_0\ \varepsilon_r$ $\quad \mu = \mu_0\ \mu' = \mu_0\ \mu_r \left(1 + j \tan \vartheta_M\right)$

$$\vec{J} = \rho \vec{v} + \sigma \vec{E}$$

IMPEDANCE IN THE (QUASI-) STATIC CASE (6/9)

- **Taking the divergence of the 3rd equation, and remembering that the divergence of a rot = 0, the continuity equation can obtained**

$$div\,\vec{J} + \frac{\partial \rho}{\partial t} = 0$$

- **Within a conductor** $\vec{J} = \sigma\,\vec{E}$ **, which using the 1st Maxwell equation yields**

$$div\,\vec{J} = \frac{\rho\,\sigma}{\varepsilon}$$

- **Using the continuity equation one finally obtains in the conductor**

$$\frac{\partial \rho}{\partial t} + \frac{\sigma}{\varepsilon}\,\rho = 0$$ **=>** $$\rho(t) = \rho(0)\,e^{-\frac{t}{\tau}}$$

with $\tau = \frac{\varepsilon}{\sigma}$ **the relaxation time of the conducting medium**

IMPEDANCE IN THE (QUASI-) STATIC CASE (7/9)

- **For perfect conductors ($\sigma = \infty$), so that the relaxation time is vanishing. For good, but not perfect, conductors it is small and of the order of 10^{-14} s or so. Therefore, for good conductors it can concluded that charges move almost instantly to the surface of the conductor. For times much larger than the relaxation time there are practically no charges inside the conductor. All of them have moved to its surface where they form a charge density Σ**

IMPEDANCE IN THE (QUASI-) STATIC CASE (8/9)

- **The situation is more involved for the magnetic field where one has to distinguish between ac and dc images**

  - **ac component**
    - **Cannot penetrate the wall of the vacuum chamber, as it was the case for the electric images**
    - **Boundary condition on the surface:** $B_{\perp} = 0$

$$\frac{F_y^{mag,ac}}{e} = -\beta^2 \, E_y \qquad \Rightarrow \qquad \frac{F_y^{ele+mag,ac}}{e} = \frac{E_y}{\gamma^2}$$

  - **dc component**
    - **Can penetrate the wall of the vacuum chamber and land on the pole faces of the magnet (if any; Not here!)**
    - **Boundary condition on the magnet poles (if any; Not here!):**

$$B_t = 0$$

IMPEDANCE IN THE (QUASI-) STATIC CASE (9/9)

- **In summary, in ~ dc (as we have no magnet poles here) the impedance contribution from the magnetic images is 0, and the wall impedance is**

With only contribution from the electric images

$$Z_x^{Wall}\left(f \sim 0 \right) = j \, \frac{L \, Z_0}{2 \pi \beta \, b^2}$$

WHY THE RESULT FROM BUROV-LEBEDEV2002 IS CLOSE TO OURS WHEREAS THEY CONSIDER ONLY TM MODES? (1/3)

- **The 6 components of the (source) electro-magnetic fields (valid for a ≤ r ≤ b, i.e. in the vacuum between the beam and the vacuum chamber) are given by**

$$E_z^{(s)}(r,\vartheta,z) = j\,C\cos\vartheta\,F_1(u)$$

$$G_z^{(s)}(r,\vartheta,z) = j\,C\sin\vartheta\,\alpha_{\mathrm{TE}}\,I_1(u)$$

$$E_\vartheta^{(s)}(r,\vartheta,z) = \gamma\,C\sin\vartheta\left[\frac{F_1(u)}{u} + \beta\,\alpha_{\mathrm{TE}}\,I_1'(u)\right]$$

$$G_\vartheta^{(s)}(r,\vartheta,z) = -\beta\,\gamma\,C\cos\vartheta\left[F_1'(u) + \frac{\alpha_{\mathrm{TE}}}{\beta}\,\frac{I_1(u)}{u}\right]$$

$$E_r^{(s)}(r,\vartheta,z) = -\gamma\,C\cos\vartheta\left[F_1'(u) + \beta\,\alpha_{\mathrm{TE}}\,\frac{I_1(u)}{u}\right]$$

$$G_r^{(s)}(r,\vartheta,z) = -\beta\,\gamma\,C\sin\vartheta\left[\frac{F_1(u)}{u} + \frac{\alpha_{\mathrm{TE}}}{\beta}\,I_1'(u)\right]$$

with

$$C = \frac{\omega\,P}{\pi\,a\,\varepsilon_0\,v^2\,\gamma^2}\,I_1(s)\,e^{-jkz}$$

$$F_1(u) = K_1(u) - \alpha_{\mathrm{TM}}\,I_1(u)$$

$$u = \frac{k\,r}{\gamma} \qquad s = \frac{k\,a}{\gamma} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$G = Z_0\,H$$

α_{TE} and α_{TM} are determined by the boundary conditions at **b and d**

WHY THE RESULT FROM BUROV-LEBEDEV2002 IS CLOSE TO OURS WHEREAS THEY CONSIDER ONLY TM MODES? (2/3)

- **For the case of 1 layer going to infinity (d→∞)**

$$\alpha_{\mathrm{TM}} = \frac{K_1(x_1)}{I_1(x_1)}\left[1 + \frac{\gamma\nu(P_1 - Q_1)(\beta x_1 x_2)^2(\gamma\nu P_1 - k\mu' Q_2)}{(\gamma\nu x_2 - k x_1)^2 - (\beta x_1 x_2)^2(\gamma\nu P_1 - k\mu' Q_2)(\gamma\nu P_1 - k\varepsilon' Q_2)}\right]$$

$$\alpha_{\mathrm{TE}} = \frac{K_1(x_1)}{I_1(x_1)} \times \frac{\gamma\nu\beta x_1 x_2(P_1 - Q_1)(\gamma\nu x_2 - k x_1)}{(\gamma\nu x_2 - k x_1)^2 - (\beta x_1 x_2)^2(\gamma\nu P_1 - k\mu' Q_2)(\gamma\nu P_1 - k\varepsilon' Q_2)}$$

with

$$x_1 = \frac{k\,b}{\gamma} \qquad x_2 = \nu\, b \qquad \nu = k\sqrt{1 - \beta^2\varepsilon'\mu'} \qquad \varepsilon = \varepsilon_0\varepsilon' = \varepsilon_0\varepsilon_r + \frac{\sigma}{j2\pi f}$$

$$\mu = \mu_0\mu' = \mu_0\mu_r\left(1 + j\tan\vartheta_M\right)$$

$$P_1 = \frac{I_1'(x_1)}{I_1(x_1)} \qquad Q_1 = \frac{K_1'(x_1)}{K_1(x_1)} \qquad Q_2 = \frac{K_1'(x_2)}{K_1(x_2)}$$

WHY THE RESULT FROM BUROV-LEBEDEV2002 IS CLOSE TO OURS WHEREAS THEY CONSIDER ONLY TM MODES? (3/3)

- **The quantity which enters in the transverse impedance is**

$$E_{\vartheta}^{(s)} + \upsilon \, B_r^{(s)} = E_{\vartheta}^{(s)} + \beta \, G_r^{(s)} = \frac{C \sin\vartheta}{\gamma} \times \frac{F_1(u)}{u}$$

=> It depends only on α_{TM} **and NOT on** α_{TE} **!**

=> This is why the results from Burov-Lebedev2002 are so close to ours (they assumed $G_z^{(s)} = 0$ **, i.e.** $\alpha_{\mathrm{TE}} = 0$ **)**

APPROXIMATE FORMULA FOR A LHC (GRAPHITE) COLLIMATOR (1/3)

- **The interesting frequency range in the LHC lies between few kHz and few GHz. In this case a simple formula can be derived for a cylindrical geometry, which should be valid for any "relatively" good conductor with real permeability and the permittivity of vacuum. It can be written as (up to a certain frequency which depends on *β* => See page 3)**

$$Z_x^{Wall}(f) = \frac{j\,L\,Z_0}{2\pi\,b^2\,\beta\,\gamma^2} + \beta\,\frac{j\,L\,Z_0}{\pi\,b^2} \times \frac{1}{1 - \dfrac{x_2}{\mu_r} \times \dfrac{K_1'(x_2)}{K_1(x_2)}}$$

with $x_2 = (1 + j)\,\dfrac{b}{\delta}$ $\delta = \sqrt{\dfrac{2}{\mu_0\,\mu_r\,\sigma\,\omega}}$

APPROXIMATE FORMULA FOR A LHC (GRAPHITE) COLLIMATOR (2/3)

- **Furthermore, this equation can be simplified even further in the two limiting cases using the following equations**

$$K_1(x) \approx 1/x \qquad I_1(x) \approx x/2$$

$$\frac{K_1'(x_2)}{K_1(x_2)} = \begin{vmatrix} -\dfrac{1}{x_2} & \text{if } |x_2| << 1 \\ -1 & \text{if } |x_2| >> 1 \end{vmatrix}$$

- **When** $|x_2| << 1$ **, i.e. at very low frequency, the transverse "wall impedance" approaches a constant inductive value**

$$Z_x^{Wall}(f \to 0) = j \frac{L Z_0}{2 \pi \beta b^2} \qquad \textbf{for} \qquad \mu_r = 1$$

APPROXIMATE FORMULA FOR A LHC (GRAPHITE) COLLIMATOR (3/3)

- When $\left| x_2 \right| >> 1$, the "classical thick-wall formula" is recovered (up to a certain frequency which depends on *β* => See page 3)

$$Z_x^{Wall}\left(f \right) = \frac{j\, L\, Z_0}{2 \pi\, b^2\, \beta\, \gamma^2} + \left(1 + j \right) \beta\, \frac{L\, Z_0\, \mu_r\, \delta}{2\, \pi\, b^3}$$

Coherent part (from the pipe) of the SC impedance

Classical thick-wall formula for the "RW" impedance

- Note that the (broad) maximum of the real part of the transverse impedance is reached when $\mathrm{Re}\left[x_2 \right] \approx 1$, i.e. $\delta \approx b$, which means

$$f_{\max,\mathrm{Re}} \approx \frac{\rho}{b^2} \times \frac{1}{\pi\, \mu_0}$$

See also picture in page 5

CONCLUSION AND FUTURE WORK

- The constant imaginary part of the impedance (when f → 0) should not belong to the RW impedance!!! It was there because we (all the people?) removed from the Total impedance the "wrong" SC impedance at low frequency (where there should be no contribution from the ac magnetic images!)
- We believe it is better to define the "wall impedance" (i.e. including also the coherent space-charge impedance) as it is not always possible to separate both contributions (PC and RW)
- Note that this new approach (understanding) has no impact on our previous predictions for LHC as $\beta = 1$!
- Future work
  - Compute the 6 EM fields in the 3 frequency regimes to have a better idea of the transition between them
  - Extend this analysis to multi-bunch beams (wave velocity ≠ bunch velocity)

=> Nicolas Mounet (new PHD student)